Name Answers

EE-3220-21 – Dr. Durant – Quiz 4
Winter 2013-'14, Week ~~3~~ 4

Given the difference equation y(n) = -0.5 y(n-1) + 0.5 x(n)

1. (2 points) Take the DTFT of both sides of the equation. Recall that the DTFT of a delayed signal, $y(n-k)$, is $e^{-j\omega k}Y(e^{j\omega})$.

$$Y(e^{j\omega}) = -0.5\,e^{-j\omega}\,Y(e^{j\omega}) + 0.5\,X(e^{j\omega})$$

2. (2 points) Solve the above equation for transfer function $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$

(2) $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{1}{2}}{1+\frac{1}{2}e^{-j\omega}} = \boxed{\frac{1}{2+e^{-j\omega}}}$

(1) $Y(e^{j\omega})\left(1+\frac{1}{2}e^{-j\omega}\right) = \frac{1}{2}X(e^{j\omega})$

3. (2 points) Let f_s = 1000 Hz, f_1 = 0 Hz, f_2 = 250 Hz, and f_3 = 500 Hz. Calculate the digital frequencies, ω_n, for each frequency, f_n. for f_1 through f_3. Recall that the digital frequency is how many radians a sinusoid moves through between samples. For example, if a signal is sampled 10 times per period, its digital frequency is 2π/10.

$\omega_1 = \frac{F_1}{F_s}2\pi = \frac{0}{1000}2\pi = 0$

$\omega_2 = \ldots \quad \frac{250}{1000}2\pi = \pi/2$

$\omega_3 = \ldots \quad \frac{500}{1000}2\pi = \pi$

4. (2 points) Evaluate H at the digital frequencies calculated above.

$H(e^{j\omega_1}) = \frac{1}{2+e^{-j0}} = \frac{1}{3}$

$H(e^{j\omega_2}) = \frac{1}{2+e^{-j\frac{\pi}{2}}} = \frac{1}{2-j} = \frac{2+j}{5}$

$H(e^{j\omega_3}) = \frac{1}{2+e^{-j\pi}} = \frac{1}{2+-1} = 1$

5. (Bonus point) Explain, using properties of the DTFT, why H at ω_3 must be a real number (its imaginary part must be 0). Recall that $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$, $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$ and, for real signals, $X(e^{-j\omega}) = X^{*}(e^{j\omega})$.

Conjugate Property : $X(e^{j\pi}) = X^{*}(e^{-j\pi})$
Periodicity Property : $X(e^{j\pi}) = X(e^{j(\pi-2\pi)}) = X(e^{-j\pi})$
Combining : $X(e^{-j\pi}) = X^{*}(e^{-j\pi}) \rightarrow$ self-conjugate $\rightarrow$ real
$X(e^{-j\pi})$ is real $\therefore$ so is $X(e^{j\pi})$

6. (2 points) Use the difference equation to complete the table below by filling in the outputs of the system when stimulated with a step input (x1) and with a causal cosine that is advancing by π radians on each sample (x2).

n	0	1	2	3	4	5	6
x1	1	1	1	1	1	1	1
y1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{5}{16}$	$\frac{11}{32}$	$\frac{21}{64}$	$\frac{43}{128}$
x2	1	-1	1	-1	1	-1	1
y2	$\frac{1}{2}$	$\frac{-3}{4}$	$\frac{7}{8}$	$\frac{-15}{16}$	$\frac{31}{32}$	$\frac{-63}{64}$	$\frac{127}{128}$

7. (Bonus point) What is the connection between your results in 4 and in 6?

$H(e^{j0}) = \frac{1}{3}$ is the DC gain. Thus it is the steady state value of y_1. y_1 is tending towards $\frac{1}{3}$ as n increases.

$H(e^{j\pi}) = 1$ is the AC ($\omega = \pi$) gain. Thus it is the steady state value of y_2. y_2 is tending toward 1 as n increases

Name Answers

EE-3220-41 – Dr. Durant – Quiz 4
Winter 2013-'14, Week ~~3~~ 4

1

Given the difference equation y(n) = 0.5 y(n-1) + 0.5 x(n)

1

1. (2 points) Take the DTFT of both sides of the equation. Recall that the DTFT of a delayed signal ~~is~~ $y(n-k)$ ~~is~~ $e^{-j\omega k}Y(e^{j\omega})$.

$$Y(e^{j\omega}) = \frac{1}{2}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{2}X(e^{j\omega})$$

2. (2 points) Solve the above equation for transfer function $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$

$$Y(e^{j\omega})\left(1-\frac{1}{2}e^{-j\omega}\right) = \frac{1}{2}X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{1}{2}}{1-\frac{1}{2}e^{-j\omega}} = \frac{1}{2-e^{-j\omega}}$$

(-1/2) N/D rearrange

3. (2 points) Let f_s = 1000 Hz, f_1 = 0 Hz, f_2 = 250 Hz, and f_3 = 500 Hz. Calculate the digital frequencies, ω_n, for each frequency, f_n. for f_1 through f_3. Recall that the digital frequency is how many radians a sinusoid moves through between samples. For example, if a signal is sampled 10 times per period, its digital frequency is 2π/10.

$$\omega_1 = \frac{f_1}{f_s}\cdot 2\pi = \frac{0}{1000}\cdot 2\pi = 0$$

$$\omega_2 = \cdots \quad \frac{250}{1000}\cdot 2\pi = \frac{\pi}{2}$$

$$\omega_3 = \cdots \quad \frac{500}{1000}\cdot 2\pi = \pi$$

4. (2 points) Evaluate H at the digital frequencies calculated above.

$-\frac{1}{4}$ per e^{-jx} error

$$H(e^{j\omega_1}) = \frac{1}{2-e^{-j0}} = \frac{1}{2-1} = 1$$

$$H(e^{j\omega_2}) = \frac{1}{2-e^{-j\pi/2}} = \frac{1}{2+j} = \frac{2-j}{5}$$

$$H(e^{j\omega_3}) = \frac{1}{2-e^{-j\pi}} = \frac{1}{2-^-1} = \frac{1}{3}$$

5. (Bonus point) Explain, using properties of the DTFT, why H at ω_3 must be a real number (its imaginary part must be 0). Recall that $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$, $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$ and, for real signals, $X(e^{-j\omega}) = X^*(e^{j\omega})$.

$H(e^{j\pi}) = H^*(e^{-j\pi})$ due to conjugate symmetry
$H(e^{j\pi}) = H(e^{j(\pi-2\pi)}) = H(e^{-j\pi})$ by 2π periodicity
Combining: $H(e^{j\pi}) = H^*(e^{j\pi}) \therefore H(e^{j\pi})$ is real
(only real #s are their own conjugate)

6. (2 points) Use the difference equation to complete the table below by filling in the outputs of the system when stimulated with a step input (x1) and with a causal cosine that is advancing by π radians on each sample (x2).

n	0	1	2	3	4	5	6
x1	1	1	1	1	1	1	1
y1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{31}{32}$	$\frac{63}{64}$	$\frac{127}{128}$
x2	1	-1	1	-1	1	-1	1
y2	$\frac{1}{2}$	$\frac{-1}{4}$	$\frac{3}{8}$	$\frac{-5}{16}$	$\frac{11}{32}$	$\frac{-21}{64}$	$\frac{43}{128}$

7. (Bonus point) What is the connection between your results in 4 and in 6?

$\lim_{n\to\infty} y_1(n) = 1 = H(e^{j\omega_1})$ DC signal is passed as predicted by H @ ω = 0

$\lim_{n\to\infty} y_2(n) = \frac{1}{3} = H(e^{j\omega_3})$ AC signal @ π rad/s is reduced in level to $\frac{1}{3}$ as predicted by H @ ω = π