

## EE3032, Winter 2019-'20, Homework 4 Solutions, Dr. Durant

**Problem 2.14:** Functions  $x(t)$  and  $h(t)$  are given by

$$x(t) = \sin(\pi t), 0 \leq t \leq 1; 0 \text{ otherwise}$$

$$h(t) = u(t)$$

Determine  $y(t) = x(t) * h(t)$ .

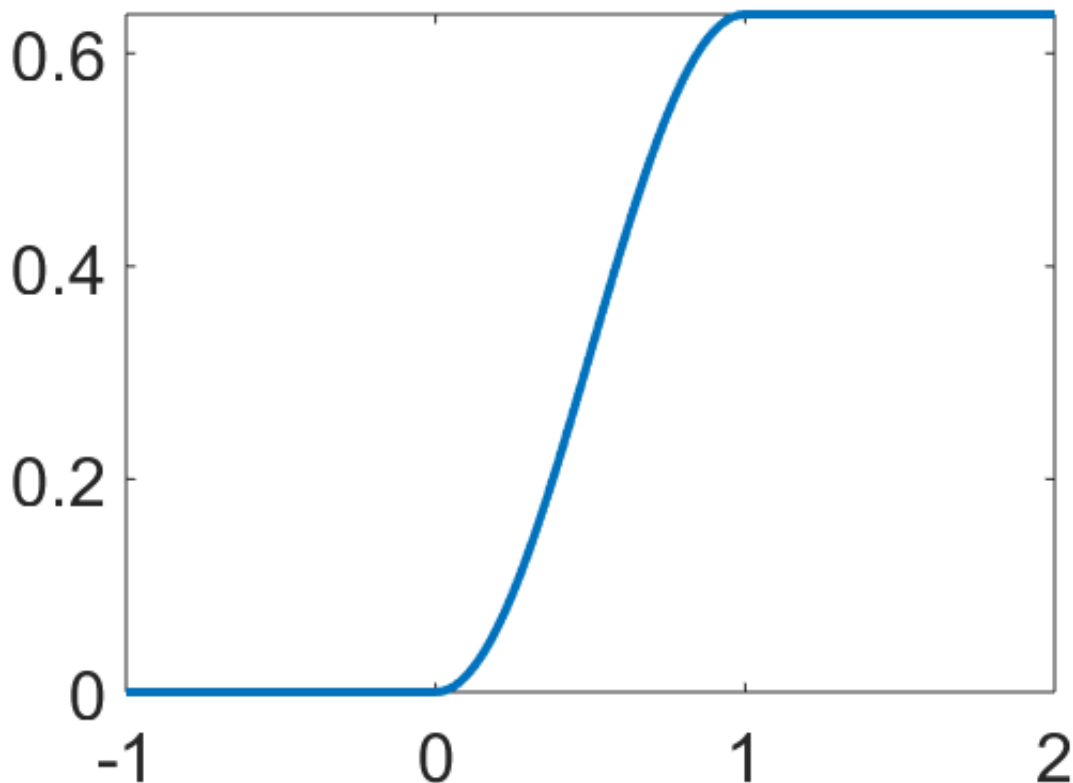
The result of convolving with  $u(t)$  is to integrate the function from  $-\infty$  to  $t$ , so the result will be 0 until  $t=0$ , vary until  $t=1$ , and then be a constant value forever.

$$y(t) = \int_0^t \sin(\pi t) dt = -\frac{1}{\pi} \cos(\pi t) \Big|_0^t = \frac{1 - \cos(\pi t)}{\pi}$$

$$y(1) = \frac{1 - \cos(\pi)}{\pi} = \frac{2}{\pi}$$

```
t = -1:0.02:2;  
y = zeros(size(t));  
r01 = 0 <= t & t <= 1;  
y(r01) = (1-cos(pi*t(r01)))/pi;  
y(t>=1) = 2/pi;  
figure, plot(t,y), title('Problem 2.14')
```

### Problem 2.14

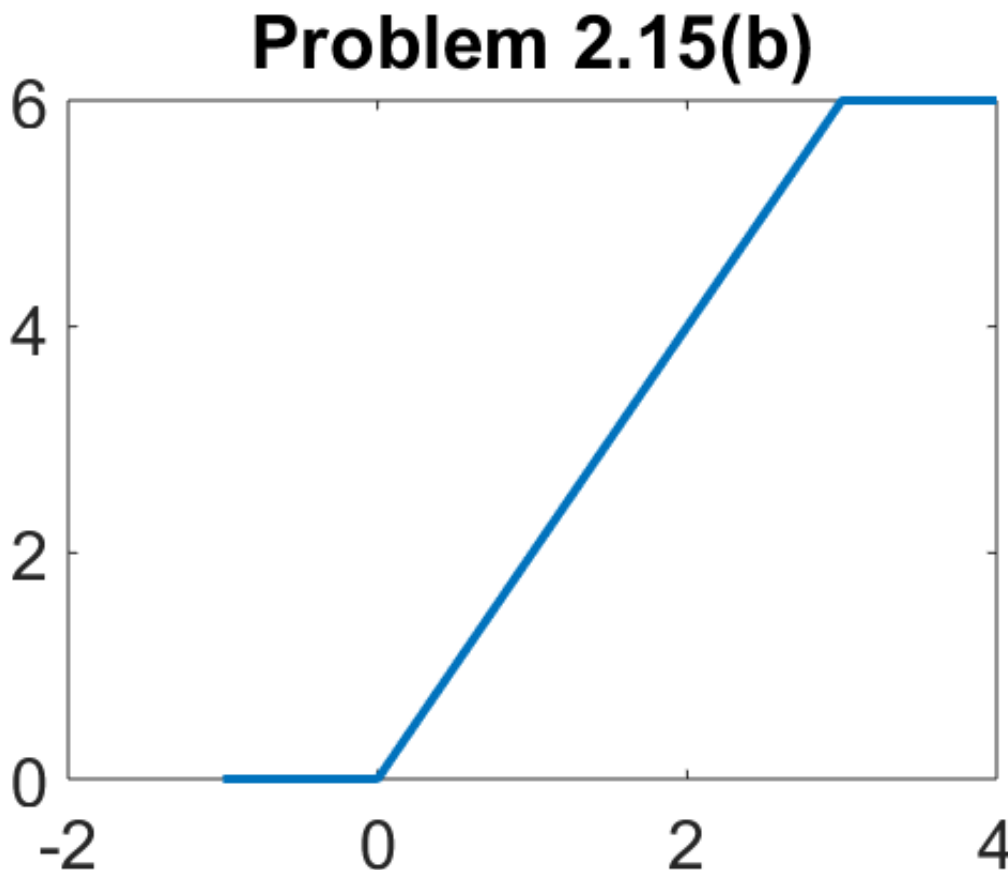


**Problem 2.15(b):** Compute the following convolutions *without computing any integrals*:  $u(t) * [2u(t) - 2u(t - 3)]$

Again, convolving with  $u(t)$  gives the integral from  $-\infty$  to  $t$  of the other function. The other function is a height 2 pulse while  $t$  is between 0 and 3 s. So, the integral is a slope 2 ramp starting at 0, and flattening off at  $t=3$ . This is accomplished by simply replacing "u" with "r" since ramp is the integral of step (is the integral of impulse).

$$y(t) = 2r(t) - 2r(t - 3)$$

```
t = -1:0.02:4;
y = zeros(size(t));
r0 = t>=0; y(r0) = 2*t(r0); % 2r(t)
r3 = t>=3; y(r3) = y(r3) - 2*(t(r3)-3); % y -= 2r(t-3)
figure, plot(t,y), title('Problem 2.15(b)')
```



**Problem 2.16(a):** Compute the following convolutions *without computing any integrals*:  $\delta(t - 2) * [u(t) - 3u(t - 1) + 2u(t - 2)]$

Consider the impulse at  $t=2$  to be the impulse response. This shows that the system just delays the input (2nd signal) by 2 s. So, we have

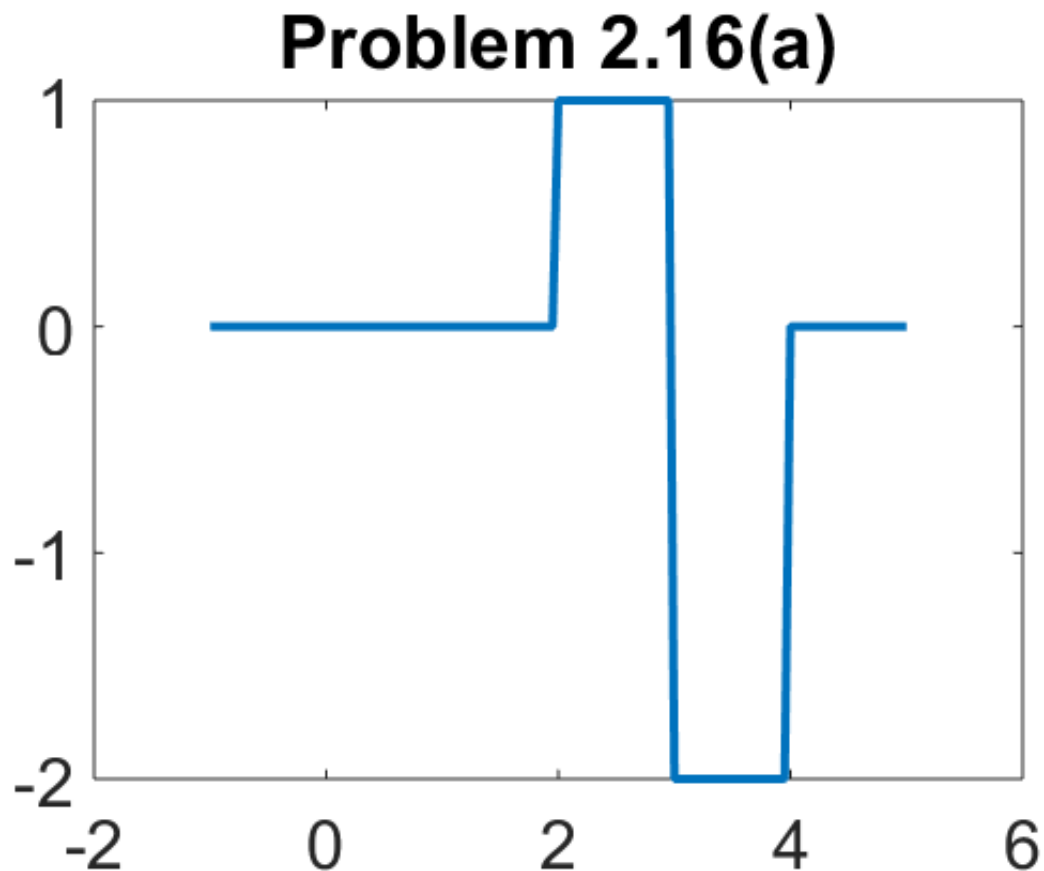
$$y(t) = u(t - 2) - 3u(t - 3) + 2u(t - 4)$$

```
t = -1:0.05:5;
y = (t>=2) - 3*(t>=3) + 2*(t>=4)
```

y = 1x121

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

```
figure, plot(t,y), title('Problem 2.16(a)')
```



**Problem 2.22(b,f)** Determine whether or not each of the LTI systems whose impulse responses are specified below are (i) causal and/or (ii) BIBO stable.

(b)  $h(t) = (1 - |t|)[u(t + 1) - u(t - 1)]$

This is **not** causal since it is non-zero for negative  $t$  due to  $u(t+1)$  starting at  $t=-1$ . This indicates outputs occur up to 1 second before inputs occur.

The system **is** stable since  $h(t)$  is absolutely integrable.  $h(t)$  is a triangle with base on  $t \in [-1, 1]$  and height 1. Therefore its (absolute) area is 1, which is finite as required for stability.

(f)  $h(t) = \frac{1}{t+1} u(t)$

This **is** causal due to multiplication by  $u(t)$  forcing a 0 value for  $t < 0$ .

The system **is not** stable. The indefinite integral is  $\ln(t+1)$ , which diverges as  $t$  approaches infinity.

**Problem 2.27: Prove the following statements.**

(a) Series combinations of BIBO-stable systems are BIBO stable.

Systems in series mean each system receives as input the output of the previous system.

Consider  $N$  stable systems. Let  $c_0$  be the finite bound on the input. Let  $A_1 \dots A_N$  be the finite maximum gain bounds, so we have outputs  $c_1$  bounded by  $c_0 A_1$ ,  $c_2$  bounded by  $c_0 A_1 A_2$ , etc. These gain bounds are multiplied to yields the bound for the composite system as  $c^* = c_1 \times c_2 \times \dots \times c_{(N-1)} \times c_N$ . We must further assume that  $N$  is finite, which is reasonable in this problem. Then  $c^*$  must be finite since a finite product of finite numbers is finite. Since  $c^*$  exists and is finite, we have a finite bound on the system output, which is the definition of BIBO stability.

(b) Series connections of causal systems are causal.

Systems in series mean each system receives as input the output of the previous system. Consider  $N$  causal systems. Let  $t_N$  be the first time for which  $h_N(t)$  is not 0; since the systems are causal,  $t_N \geq 0$  for all  $N$ . We proved in class that we can get the impulse response of a composite of systems in series by convolving all the impulse responses. We also showed, via the width property, that the the first time for which the result of convolution is non-zero is the sum of when each of the inputs is non-zero. By induction, we can then say that if we convolve  $N$  functions the first time that the result is non-zero is the sum of all the  $t_N$ . All  $t_N$  are  $\geq 0$ . Therefore the sum of all  $t_N$  is  $\geq 0$  and the result is causal. (Note: in this case, we did **not** need to require that there be a finite number of systems.)

**Problem 2.29(c,d,f) An LTI system has the frequency response function**

$$H(\omega) = \frac{1}{j\omega + 3}. \text{ Compute the output if the inputs is}$$

(c)  $x(t) = 5 \cos(4t)$

Calculate the gain and phase shift:  $H(4) = \frac{1}{j4 + 3} = 0.2 \angle -0.9273$

$y(t) = 5 \times 0.2 \cos(4t - 0.9273) = \cos(4t - 0.9273)$  [-0.9273 =  $\text{atan2}(-4,3)$  = -53 degrees]

(d)  $x(t) = \delta(t)$

Recognize that  $h(t) = e^{-3t}u(t)$  yields the given transfer function. Then,  $h(t) = y(t)$  since it is given that the input is the unit impulse.

Note: Common errors on this problem were assuming that  $x(t)$  has a single frequency (it doesn't; we will show later that it actually has an infinite number of frequencies) and that due to its infinitely narrow width that the output is 0 (The delta still has area 1 (often volts times seconds)).

(f)  $x(t) = 1$

$$H(0) = \frac{1}{3}$$

$y(t) = 1/3$

**Problem 2.31: An LTI system has the impulse response  $h(t) = e^{-12t}u(t)$ . The input to the system is  $x(t) = 12 + 26 \cos(5t) + 45 \cos(9t) + 80 \cos(16t)$ . Compute the output  $y(t)$ .**

Recognize that the corresponding  $H(\omega) = \frac{1}{j\omega + 12}$ . And we have 4 components (including DC) with...

```
omega = [0 5 9 16];  
A = [12 26 45 80];  
H = 1./ (1j*omega+12);  
disp([abs(H); angle(H); rad2deg(angle(H))]) % show in polar form
```

```
0.0833    0.0769    0.0667    0.0500  
0    -0.3948   -0.6435   -0.9273  
0   -22.6199  -36.8699  -53.1301
```

```
Y = A.*abs(H); % amplitude = original amplitude times gain  
disp(Y)
```

```
1.0000    2.0000    3.0000    4.0000
```

All the cosines were input with no phase shift, so the phase shifts due to H are the resulting phases. Putting it all together...

$$y(t) = 1 + 2 \cos(5t - 0.3948) + 3 \cos(9t - 0.6435) + 4 \cos(16t - 0.9273)$$