FOURIER SERIES

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References: Alexander and Sadiku, Chapter 17; Chaparro, Chapter 4

TRIGONOMETRIC FOURIER SERIES

Fourier discovered that any periodic signal $x(t)$, satisfying certain (Dirichlet) conditions, could be written as a sum of sinusoids having frequencies that are integer multiples of the fundamental frequency $\omega_0 = 2\pi f_0 = 2\pi/T_0$:

$$x(t) = c_0 + \sum_{k=1}^{\infty}[c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)],$$

where the coefficients $\{c_k\}|_{k=0}^{\infty}, \{d_k\}|_{k=1}^{\infty}$ can be found by projecting onto orthogonal sinusoidal basis functions:

Even (cosine-based) terms for integers $k \geq 0$:

$$\langle x(t), \cos(k\omega_0 t)\rangle = \frac{1}{T_0}\int_{\langle T_0\rangle} x(t)\cos(k\omega_0 t)\,dt = \frac{1}{T_0}\int_{\langle T_0\rangle}\{c_0 + \sum_{m=1}^{\infty}[c_m \cos(m\omega_0 t) + d_m \sin(m\omega_0 t)]\}\cos(k\omega_0 t)\,dt$$

$$= \frac{1}{T_0}\int_{\langle T_0\rangle} c_0 \cos(k\omega_0 t)\,dt + \sum_{m=1}^{\infty} c_m \frac{1}{T_0}\int_{\langle T_0\rangle}\cos(m\omega_0 t)\cos(k\omega_0 t)\,dt$$

$$+ \sum_{m=1}^{\infty} d_m \frac{1}{T_0}\int_{\langle T_0\rangle}\sin(m\omega_0 t)\cos(k\omega_0 t)\,dt$$

Due to the orthogonality of the sine and cosine basis functions, recall that

$$\frac{1}{T_0}\int_{\langle T_0\rangle}\cos(m\omega_0 t)\cos(k\omega_0 t)\,dt = \langle\cos(m\omega_0 t), \cos(k\omega_0 t)\rangle = \begin{cases} 1, & m = k = 0 \\ \frac{1}{2}, & m = k > 0, \\ 0, & m \neq k \end{cases}$$

$$\frac{1}{T_0}\int_{\langle T_0\rangle}\sin(m\omega_0 t)\sin(k\omega_0 t)\,dt = \langle\sin(m\omega_0 t), \sin(k\omega_0 t)\rangle = \begin{cases} 1, & m = k = 0 \\ \frac{1}{2}, & m = k > 0, \\ 0, & m \neq k \end{cases}$$

$$\frac{1}{T_0}\int_{\langle T_0\rangle}\sin(m\omega_0 t)\cos(k\omega_0 t)\,dt = \langle\sin(m\omega_0 t), \cos(k\omega_0 t)\rangle = 0.$$

Hence the above term simplifies to

$$\langle x(t), \cos(k\omega_0 t)\rangle = c_0\delta[k] + \sum_{m=1}^{\infty} c_m \frac{1}{2}\delta[m-k] + \sum_{m=1}^{\infty} d_m \cdot 0 = \begin{cases} c_0 & k = 0 \\ \frac{c_k}{2} & k \geq 1 \end{cases}.$$

Hence the "c" coefficients can be obtained as

$$c_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) dt$$

$$c_k = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \cos(k\omega_0 t)\, dt\,, k \geq 1.$$

In a similar fashion, the odd (sine-based) terms can be gotten for $k \geq 1$:

$$\langle x(t), \sin(k\omega_0 t) \rangle = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) \sin(k\omega_0 t)\, dt = \frac{1}{T_0} \int_{\langle T_0 \rangle} \{c_0 + \sum_{m=1}^{\infty} [c_m \cos(m\omega_0 t) + d_m \sin(m\omega_0 t)]\} \sin(k\omega_0 t)\, dt$$

$$= \frac{1}{T_0} \int_{\langle T_0 \rangle} c_0 \sin(k\omega_0 t)\, dt + \sum_{m=1}^{\infty} c_m \frac{1}{T_0} \int_{\langle T_0 \rangle} \cos(m\omega_0 t) \sin(k\omega_0 t)\, dt$$

$$+ \sum_{m=1}^{\infty} d_m \frac{1}{T_0} \int_{\langle T_0 \rangle} \sin(m\omega_0 t) \sin(k\omega_0 t)\, dt$$

Using orthogonality, observe that

$$\langle x(t), \sin(k\omega_0 t) \rangle = c_0 0 + \sum_{m=1}^{\infty} c_m \cdot 0 + \sum_{m=1}^{\infty} d_m \frac{1}{2} \delta[m-k] = \frac{d_k}{2}.$$

Hence the "d" coefficients can be obtained as

$$d_k = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \sin(k\omega_0 t)\, dt.$$

In summary, the trigonometric Fourier series coefficients can be calculated as follows:

Average value (also known as DC term):

$$c_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) dt$$

Cosine amplitude at frequency $k\omega_0$, $k \geq 1$:

$$c_k = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \cos(k\omega_0 t)\, dt$$

Sine amplitude at frequency $k\omega_0$, $k \geq 1$:

$$d_k = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \sin(k\omega_0 t)\, dt$$

EXPONENTIAL FOURIER SERIES (EFS) FROM TRIG FS

Another important form of the Fourier series is the exponential Fourier series, where complex exponentials form the basis functions for periodic signals in this space:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}.$$

The EFS coefficients $\{X_k\}$ are complex numbers whose magnitude signifies the strength of the signal at frequency $k\omega_0$ and whose phase angle is the phase shift (relative to cosine) at the same frequency.

These coefficients can be determined by projecting the signal onto the corresponding complex exponential:

$$X_k = \langle x(t), e^{jk\omega_0 t} \rangle = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-jk\omega_0 t} dt.$$

To compare the EFS to the trig FS form, first consider the trig FS form:

$$x(t) = c_0 + \sum_{k=1}^{\infty} [c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)].$$

Using Euler's identity:

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[\frac{c_k}{2} (e^{jk\omega_0 t} + e^{-jk\omega_0 t}) + \frac{d_k}{2j} (e^{jk\omega_0 t} - e^{-jk\omega_0 t}) \right].$$

Distributing and putting together similar terms:

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[\left(\frac{c_k}{2} + \frac{d_k}{2j} \right) e^{jk\omega_0 t} + \left(\frac{c_k}{2} - \frac{d_k}{2j} \right) e^{-jk\omega_0 t} \right].$$

Observing that $\frac{1}{j} = -j$:

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[\frac{(c_k - jd_k)}{2} e^{jk\omega_0 t} + \frac{c_k + jd_k}{2} e^{-jk\omega_0 t} \right].$$

Splitting into positive and negative k terms:

$$x(t) = c_0 + \sum_{k=1}^{\infty} \frac{(c_k - jd_k)}{2} e^{jk\omega_0 t} + \sum_{k=1}^{\infty} \frac{c_k + jd_k}{2} e^{-jk\omega_0 t}.$$

Changing indices for the negative k values:

$$x(t) = c_0 + \sum_{n=1}^{\infty} \frac{(c_k - jd_k)}{2} e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} \frac{c_{-k} + jd_{-k}}{2} e^{jk\omega_0 t}.$$

Now denote, for $k \geq 1$:

$$X_k = \frac{c_k - jd_k}{2}$$

$$X_{-k} = \frac{c_k + jd_k}{2} = X_k^*.$$

Also denote $X_0 = c_0$. Then

$$x(t) = X_0 + \sum_{k=1}^{\infty} X_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} X_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$$

is now in EFS form.

Note that the EFS form can be obtained for any complex-valued periodic signal satisfying Dirichlet conditions, whereas the trig FS form is typically only used on real-valued signals, as has been assumed in the previous derivation.

Also note that the EFS coefficients can be used to plot a two-sided frequency spectrum: the magnitude $|X_k|$ and phase $\angle X_k$ can be used to represent the magnitude and phase at nth harmonic frequency $k\omega_0$, where in this case, k can be a negative integer.

For a real-valued signal $x(t)$, observe that the magnitude $|X_{-k}| = |X_k|$ will be even-symmetric and the phase $\angle X_{-k} = -\angle X_k$ will be odd-symmetric.

Example: Consider the signal

$$x(t) = 10\cos(400\pi t) - 7\sin(700\pi t) + 4\sin(900\pi t).$$

Determine the nonzero EFS coefficients for the signal in order to plot the two-sided magnitude and phase spectra for the signal.

Appendix

Compact trigonometric Fourier series (aka cosine FS or amplitude-phase form)

The compact trig FS is a way to combine the cosine and sine amplitudes at a single harmonic frequency $n\omega_0$ into equivalent amplitude and phase values for a cosine at the same frequency.

Observe that

$$c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) = \sqrt{c_k^2 + d_k^2} \cos\left(k\omega_0 t - tan^{-1}\left(\frac{d_k}{c_k}\right)\right).$$

For $k \geq 1$, denote

$$A_k = \sqrt{c_k^2 + d_k^2}$$

to be the cosine amplitude and

$$\theta_k = -\tan^{-1}\left(\frac{d_k}{c_k}\right)$$

to be the phase at the kth harmonic frequency $k\omega_0$. Also denote $A_0 = c_0$ to be the average value (or DC term).

We can now write any periodic signal $x(t)$ (satisfying Dirichlet conditions) in compact trig FS form:

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k).$$

The amplitude-phase form of the FS is useful for plotting the one-sided frequency spectrum of $x(t)$. Recall that A_k and θ_k represent the cosine magnitude and phase, respectively, at the kth harmonic frequency $k\omega_0$.

Fourier Series Conversions

For all integers $k \geq 1$, the EFS coefficients may be obtained from the cosine or trig FS coefficients as follows:

$$X_k = \frac{A_k}{2}\angle\theta_k = \frac{\sqrt{c_k^2 + d_k^2}}{2}\angle\theta_k = \frac{c_k - jd_k}{2}.$$

The magnitude of the EFS coefficient is

$$|X_k| = \frac{A_k}{2} = \frac{\sqrt{c_k^2 + d_k^2}}{2}.$$

One exception – the DC term: $X_0 = A_0 = c_0$

The phase angle of the EFS coefficient is the same as the angle of the cosine Fourier series:

$$\angle X_k = \theta_k = -\tan^{-1}\left(\frac{d_k}{c_k}\right).$$

Summary table:

From/to	**cosine FS**	**trig FS**	**EFS**
cosine FS		$c_k = A_k \cos\theta_k$ $d_k = -A_k \sin\theta_k$	$X_k = \frac{A_k}{2}\angle\theta_k$ $X_0 = A_0$
trig FS	$A_k = \sqrt{c_k^2 + d_k^2}$ $\theta_k = -\tan^{-1}\left(\frac{d_k}{c_k}\right)$		$X_k = \frac{c_k - jd_k}{2}$ $X_0 = c_0$
EFS	$A_0 = X_0$; $A_k = 2\|X_k\|$, $\theta_k = \angle X_k$	$c_0 = X_0$; $c_k = 2\, Re\{X_k\}$ $d_k = -2\, Im\{X_k\}$	