Transformations

- Based on homogeneous coordinates
  - Actual Point Homogeneous Coord.

$$\vec{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \vec{P}_h = \begin{bmatrix} x_h \\ y_h \\ z_h \\ h \end{bmatrix}$$

- $x = x_h/h \quad y = y_h/h \quad z = z_h/h$

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Translation

- Move in x, y, and z directions
  - $x' = x + T_x$, $y' = y + T_y$, $z' = z + T_z$, $h' = h$

$$\vec{T} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \vec{P}'_h = \vec{T}\vec{P}_h$$

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Scaling

- Use translation to scale about any point

$$\vec{P}'_h = \vec{T}(x_f, y_f, z_f)\,\vec{S}(s_x, s_y, s_z)\,\vec{T}(-x_f, -y_f, -z_f)\,\vec{P}_h$$

$$\vec{S}(S_x, S_y, S_z) = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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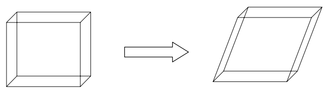
Shearing



- Shear in x based on y coordinate

$$\vec{Sh}(sh_{xy}) = \begin{bmatrix} 1 & sh_{xy} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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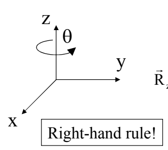
Rotation (1)

- There are an infinite # of rotation axes
- Simplest cases first
  - Ex. CCW about z-axis



$$\vec{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$
$$h' = h$$

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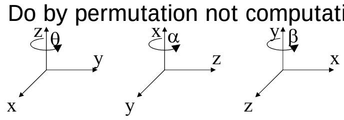
Rotation (2)

- Rotation about x-axis and y-axis
  - Do by permutation not computation



$$\vec{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \vec{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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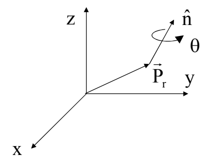
Arbitrary Rotation (1)

- About point and axis

$$\vec{P}_r = \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix}$$

$$\hat{n} = \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix};\ x_n^2 + y_n^2 + z_n^2 = 1$$



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Arbitrary Rotation (2)

$$\vec{R}_{\hat{n}}(\theta) = \vec{T}(\vec{P}_r)\vec{R}_x(-\alpha)\vec{R}_y(-\beta)\vec{R}_z(\theta)\vec{R}_y(\beta)\vec{R}_x(\alpha)\vec{T}(-\vec{P}_r)$$

$$\cos\alpha = \frac{z_n}{\sqrt{y_n^2+z_n^2}} \qquad \cos\beta = \sqrt{y_n^2+z_n^2}$$

$$\sin\alpha = \frac{y_n}{\sqrt{y_n^2+z_n^2}} \qquad \sin\beta = -x_n$$

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3-D Viewing

1. Local — Modeling Xform: Model → World
   - Draw the picture
2. Global — Viewing Xform: World → View
   - Observer position
3. Global — Clip
   - Remove objects which can't be seen
4. Global — Projection Xform: View → Projection
   - 3D → 2D
5. Global — Workstation Xform: Projection → Device
   - Adjust size and position on screen

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Viewing Coordinates (1)

- Convenient reference for viewing
- Account for observer
  - Position
  - Orientation
    - Direction of gaze
    - Which direction is "up"

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Viewing Coordinates (2)

1. Define a plane whose normal vector is in the view direction: $\mathbf{z_v} = \mathbf{N}$
   - "The film"
2. Determine if viewing is relative to a point $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$
   - Should we clip behind the camera?
3. Select a view-up vector **v** that is not parallel to $\mathbf{z_v}$ that will specify $\mathbf{y_v}$

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Transformation Process (1)

1. Translate the world to the view origin
   - $\mathbf{T(-x_0, -y_0, -z_0)}$
2. Rotate the world coordinate system to align with the viewing coordinate system
   - Make three rotations: $\mathbf{R_x, R_y, R_z}$
     - A reflection may also be needed
   - Or make a basis transformation

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Transformation Process (2)

$$\hat{n} = \frac{\vec{N}}{|\vec{N}|} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \hat{z}_v$$

$$\hat{u} = \frac{\vec{V} \times \vec{N}}{|\vec{V} \times \vec{N}|} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \hat{x}_v$$

$$\hat{v} = \hat{n} \times \hat{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \hat{y}_v$$

$$\vec{R} = \begin{bmatrix} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vec{M}_{wc,vc} = \vec{R}\vec{T}$$

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Projection Transformation

- Moving 3-D viewed scene to 2-D window
- Two types
  - Parallel
    - Viewed size is not a function of distance
  - Perspective
    - Viewed size is a function of distance

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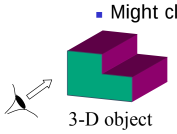
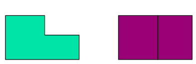
Orthographic Projection

- Project parallel to the direction of view
  - E.g. front, side, and plan views
  - Aligned axes make this easy
    - Might clip for $z_v < 0$



3-D object



Two possible orthographic views

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & & & \\ & & -1 & \\ & & & \\ & & & 1 \end{bmatrix}$$

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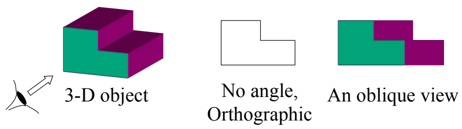
Oblique Projection

- Project at angle to the direction of view
  - Like looking out of the side of our eye



$$\begin{bmatrix} 1 & & a \\ & 1 & \\ & & 1 \end{bmatrix}$$

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Axonometric Projection (1)

- All 3 axes projected onto x-y plane
  - Parallel lines still remain parallel
  - Generalized extension to oblique
  - View "all" 3 sides at a time
- Isometric
  - All 3 units have same projected length
- Dimetric
  - Exactly 2 units have the same projected length
- Typically up (y) remains up

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Axonometric Projection (2)

- Common Isometric Formats
  - 30° – x & z 30° above horizontal
    - All axes 120° apart
  - 1:2 – x & z rise by 1 for run of 2
    - (Bresenham?) (atan(1/2) = 26.5...°)
  - Military – x & z 45° above horizontal
    - separated by 90°, so building tops (x-z plane) not skewed
- Dimetric Example (x-y vs. z)

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Axonometric Projection (3)

- General, with y upward
- Isometric (all projected lengths equal)

$$\sqrt{a^2+c^2} = d = \sqrt{b^2+e^2}$$

- Dimetric?
- General?

$$\begin{bmatrix} a & & b & \\ c & d & e & \\ & & & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & & -\cos\theta & \\ \sin\theta & 1 & \sin\theta & \\ & & & \\ & & & 1 \end{bmatrix}$$

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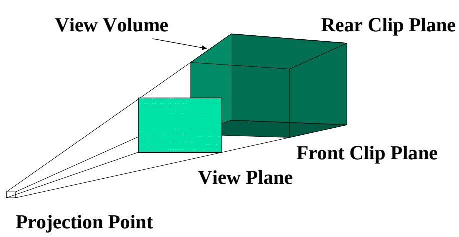
Perspective Projection (1)



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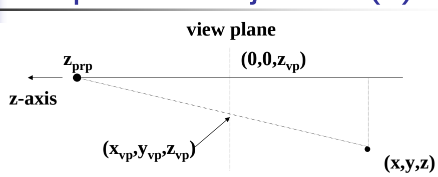
Perspective Projection (2)



$$\frac{x}{z - z_{prp}} = \frac{x_{vp}}{z_{vp} - z_{prp}}$$

$$x_{vp} = \frac{x(z_{vp} - z_{prp})}{z - z_{prp}}$$

$$\vec{M}_{projection} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{z_{vp}-z_{prp}} & \frac{-z_{prp}}{z_{vp}-z_{prp}} \end{bmatrix}$$

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