3-D Transformations

- Similar to 2-D case
- Homogeneous coordinates
  - 4-element vector (x_h, y_h, z_h, h)
- Transformation matrices
  - 4x4 matrix for each transformation
  - Translation, rotation, etc.

1

Translation

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} = \begin{bmatrix} x + T_x \\ y + T_y \\ z + T_z \\ 1 \end{bmatrix}$$

2

Scaling

$$\mathbf{S} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} = \begin{bmatrix} xS_x \\ yS_y \\ zS_z \\ 1 \end{bmatrix}$$

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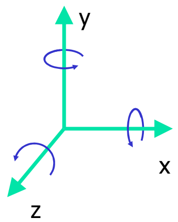
3-D Rotation About an Axis



Positive rotation is counterclockwise, when looking from positive direction along an axis.

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Rotation About z-Axis

$$\mathbf{R}_z(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad \mathbf{P}=\begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix}$$

$$\mathbf{P}'=\mathbf{R}_z(\theta)\cdot\mathbf{P}=\begin{bmatrix}?\\ ?\\ ?\\ ?\end{bmatrix}=\begin{bmatrix}x\cos\theta-y\sin\theta\\ x\sin\theta+y\cos\theta\\ z\\ 1\end{bmatrix}$$

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Rotation About x-Axis

$$\mathbf{R}_x(\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad \mathbf{P}=\begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix}$$

$$\mathbf{P}'=\mathbf{R}_x(\theta)\cdot\mathbf{P}=\begin{bmatrix}?\\ ?\\ ?\\ ?\end{bmatrix}=\begin{bmatrix}x\\ y\cos\theta-z\sin\theta\\ y\sin\theta+z\cos\theta\\ 1\end{bmatrix}$$

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Generalized 3-D Rotations

- Rotation around any axis
  - Not necessarily a coordinate axis
- Procedure
  - Similar to 2-D case
  - Composite translation and rotation transformation

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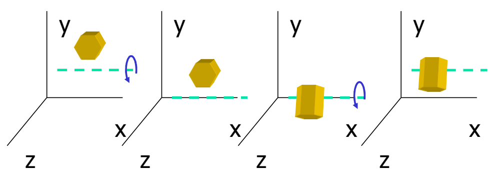
Special Case: Parallel Axis

Rotation axis parallel to a coordinate axis.



Translate to axis. Rotate. Translate back.

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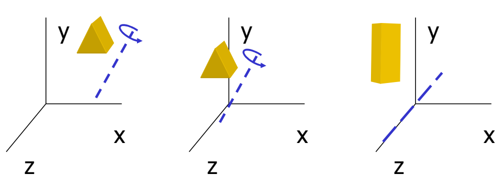
Rotation Around Any Axis (1)



Translate to origin. Rotate to z axis.

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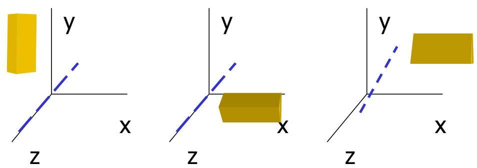
Rotation Around Any Axis (2)



Rotate around z axis.

Rotate axis back.

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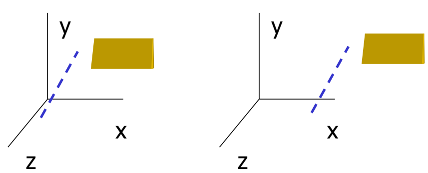
Rotation Around Any Axis (3)



Translate back to original axis position.

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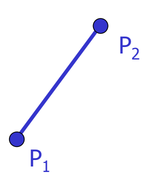
General Rotation (1)



Rotation axis determined by P_1 and P_2.

Translate P_1 to origin.

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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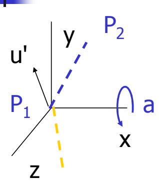
General Rotation (2)



$$\mathbf{U} = \mathbf{P}_2 - \mathbf{P}_1$$

$$\mathbf{u} = (a, b, c)$$

$$\mathbf{u}' = (0, b, c)$$

$$\mathbf{u}' \cdot \mathbf{u}_z = |\mathbf{u}'||\mathbf{u}_z| \cos\alpha$$

$$\cos\alpha = \frac{\mathbf{u}' \cdot \mathbf{u}_z}{|\mathbf{u}'||\mathbf{u}_z|} = \frac{c}{d}$$

$$d = \sqrt{b^2 + c^2}$$

$$\mathbf{u}' \times \mathbf{u}_z = \mathbf{u}_x |\mathbf{u}'||\mathbf{u}_z| \sin\alpha = \mathbf{u}_x b$$

$$d \sin\alpha = b$$

$$\sin\alpha = \frac{b}{d}$$

Calculated cross product

Rotate around x-axis, into xz plane.
Need sin α and cos α for rotation matrix.

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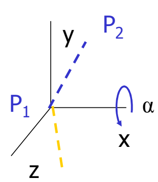
General Rotation (3)



$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotate around x-axis, into xz plane.

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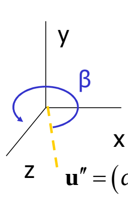
General Rotation (4)



$$\mathbf{u}'' \cdot \mathbf{u}_z = |\mathbf{u}''||\mathbf{u}_z| \cos\beta$$

$$\cos\beta = \frac{\mathbf{u}'' \cdot \mathbf{u}_z}{|\mathbf{u}''||\mathbf{u}_z|} = d$$

$$\mathbf{u}'' = (a, 0, d)$$

$$\mathbf{u}'' \times \mathbf{u}_z = \mathbf{u}_y |\mathbf{u}''||\mathbf{u}_z| \sin\beta = \mathbf{u}_y(-a)$$

$$\sin\beta = -a$$

Rotate around y-axis, onto z-axis.

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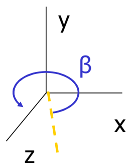
General Rotation (5)



$$\mathbf{R}_y(\beta) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotate around y-axis, onto z-axis.

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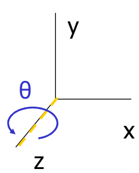
General Rotation (6)



$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now apply actual desired rotation (θ) around z-axis.

Next step is to reverse the "setup" rotations and translation.

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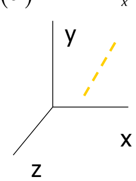
General Rotation (7)

$$\mathbf{R}(\theta) = \mathbf{T}^{-1} \cdot \mathbf{R}_x^{-1}(\alpha) \cdot \mathbf{R}_y^{-1}(\beta) \cdot \mathbf{R}_z(\theta) \cdot \mathbf{R}_y(\beta) \cdot \mathbf{R}_x(\alpha) \cdot \mathbf{T}$$



Translate to origin
Rotate around x-axis
Rotate around y-axis
Rotate as specified around z-axis
Reverse y-axis rotation
Reverse x-axis rotation
Reverse translation

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