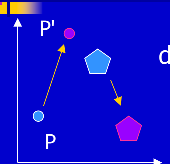
Transformations

- Translation
- Rotation
- Scaling
- Reflection
- Shear

1

Translation



Move point by adding translation distances, t_x and t_y, to coordinates.

$$\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

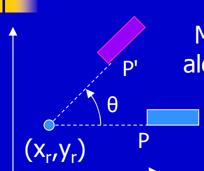
$$\mathbf{P}' = \mathbf{P} + \mathbf{T} = \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix}$$

Move other shapes by translating all point coordinates.

Translation is a "rigid body" transformation.

2

Rotation



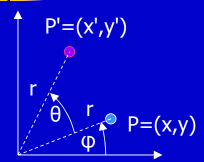
Move point(s) by repositioning along circular arc(s) centered on the rotation point (x_r, y_r).

New position is such that the angle from the rotation point to each rotated point increases by θ. Positive rotation is counter-clockwise.

Rotation is also a "rigid body" transformation.

3

Rotation Point at Origin



Easiest to work in polar coordinates

$$\mathbf{P} = (x, y) = r\angle\phi$$
$$x = r\cos\phi$$
$$y = r\sin\phi$$

$$\mathbf{P}' = (x', y') = r\angle(\phi+\theta)$$
$$x' = r\cos(\phi+\theta)$$
$$y' = r\sin(\phi+\theta)$$

4

Trigonometry Flashback

$$\cos(\phi+\theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$
$$\sin(\phi+\theta) = \cos\phi\sin\theta + \sin\phi\cos\theta$$

$$r\cos(\phi+\theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta$$
$$r\sin(\phi+\theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta$$

But:

$$x = r\cos\phi \qquad x' = r\cos(\phi+\theta)$$
$$y = r\sin\phi \qquad y' = r\sin(\phi+\theta)$$

So:

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

5

Rotation Matrix

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

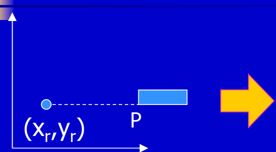
$$\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{P}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

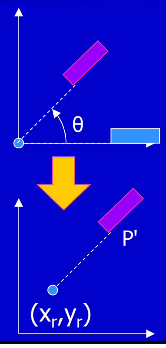
$$\mathbf{P}' = \mathbf{R}\cdot\mathbf{P}$$

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

6

Rotation Around a Point





1) Translate rotation point to origin
2) Rotate around origin
3) Translate back

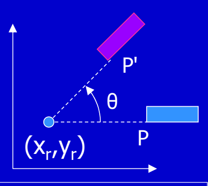
7

Full Rotation Formula

$$\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{P}_r = \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$

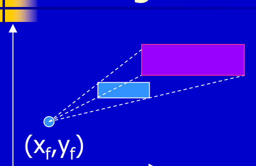
$$\mathbf{P}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$



$$\mathbf{P}' = \mathbf{R} \cdot (\mathbf{P} - \mathbf{P}_r) + \mathbf{P}_r$$

8

Scaling

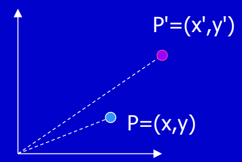


Move points by repositioning so distance from a fixed point (x_r, y_r) is multiplied by a factor (s).

Can have uniform ($s_x = s_y$) or differential ($s_x \neq s_y$) scaling.

9

Fixed Point at Origin



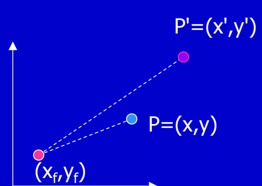
$$x' = x \cdot s_x$$
$$y' = y \cdot s_y$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

$$\mathbf{S} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

10

Scaling with General Fixed Point



$$\mathbf{S} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \left(\mathbf{P} - \mathbf{P}_f\right) + \mathbf{P}_f$$

11

Matrix Representations

- Assume series of transformations
  - Translate, rotate, scale, etc.
- Can apply them in sequence
  - Multiple matrix calculations
- Is there a better way?

12

Homogeneous Coordinates

- Normal 2-D Cartesian
  - Two coordinate values
    - x, y
- Homogeneous
  - Three coordinate values
    - x_h, y_h, h
    - Often choose h=1

$$x = \frac{x_h}{h}$$

$$y = \frac{y_h}{h}$$

13

Homogeneous?

- Remember from MA-235 (DiffEq)?
  - Homogeneous polynomial
    - All terms of same degree (*e.g.*, x^4y+7y^5)
  - Line:
    - Ax + By + C = 0
      - mixed degree therefore non-homogeneous
    - $Ax_h/h + By_h/h + C = 0$
    - $Ax_h + By_h + Ch = 0$
      - homogeneous polynomial of degree 1 in (x, y, h)
- Transformation representation
  - All in matrix form

14

Matrix Form of Transformation

$$\mathbf{P} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{P}$$

$$\mathbf{T} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

15

Matrix Example 1

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

16

Matrix Example 2

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

Recognize this result?

17

Matrix Example 3

$$\mathbf{T} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

How about this one?

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ 1 \end{bmatrix}$$

18

Matrix Example 4

Projection of (1,0,0) – x unit

$$\mathbf{T} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} = \begin{bmatrix} ax \\ by \\ 1 \end{bmatrix}$$

And this one?

19

Matrix Example 5

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} = \begin{bmatrix} x \\ -y \\ 1 \end{bmatrix}$$

20

Matrix Example 6

$$\mathbf{T} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} = \begin{bmatrix} x + ay \\ y \\ 1 \end{bmatrix}$$

21

Composite Transformations

- Each transformation
  - Represented by a 3x3 matrix
  - Applied by matrix multiplication
- Can combine transformations
  - By multiplying matrices
- Full transformation in one step!

ABCDx = A(B(C(Dx))) = (((AB)C)D)x

22

Composite Example 1

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{T}_2 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{T}_1 \cdot \mathbf{T}_2 \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a+c \\ y+b+d \\ 1 \end{bmatrix}$$

23

Summary

- Transformation representation
  - Represent any transform by 3x3 matrix
    - Equivalent to 2x2 separate operations
      - Addition, multiplication
  - Composite formed by matrix product
    - Can't do this with 2x2 operations!
- Apply to each point
  - To yield transformed point
- Try it yourself!

24