Math Review

- Coordinate systems
  - 2-D, 3-D
  - Rectangular, polar
- Vectors
- Matrices
  - Matrix operations

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Math Review

- Cornerstone of graphics
- Basis for most algorithms
- Systematic notation
  - Simplifying communication
  - Organizing ideas
  - Compact representation

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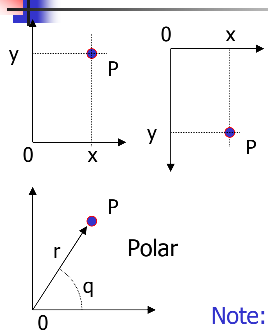
2-D Coordinate Systems



Rectangular

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Note: `atan()` versus `atan2()`

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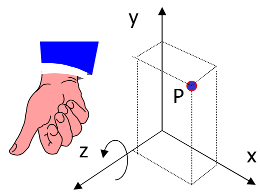
3-D Coordinate Systems

Right-hand rule



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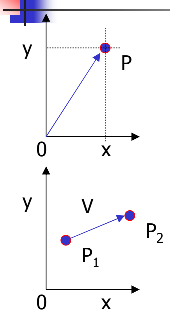
Points and Vectors



Point (x,y) can be represented by vector (V_x, V_y) from origin.

In general, vector represents difference (directed distance) between two points.

$$\mathbf{V} = \mathbf{P}_2 - \mathbf{P}_1 = \left(x_2 - x_1, y_2 - y_1\right) = \left(V_x, V_y\right)$$

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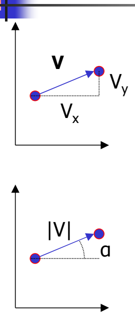
2-D Vector Representations



Cartesian components

$$\mathbf{V} = \left(V_x, V_y\right)$$

Magnitude and direction angle

$$|\mathbf{V}| = \sqrt{V_x^2 + V_y^2}$$

$$\alpha = \tan^{-1}\left(\frac{V_y}{V_x}\right)$$

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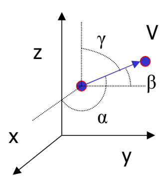
3-D Vector Representations

Cartesian components $\mathbf{V} = \left(V_x, V_y, V_z\right)$

Magnitude and direction cosines



$$|\mathbf{V}| = \sqrt{V_x^{\,2} + V_y^{\,2} + V_z^{\,2}}$$

$$\cos\alpha = \frac{V_x}{|\mathbf{V}|}, \cos\beta = \frac{V_y}{|\mathbf{V}|}, \cos\gamma = \frac{V_z}{|\mathbf{V}|}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

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Vector Operations

Addition $\mathbf{V}_1 + \mathbf{V}_2 = \left(V_{1_x} + V_{2_x}, V_{1_y} + V_{2_y}, V_{1_z} + V_{2_z}\right)$

Scalar multiply $a\mathbf{V} = \left(aV_x, aV_y, aV_z\right)$

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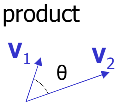
Vector Operations

Inner (dot) product

$$\mathbf{V}_1 \bullet \mathbf{V}_2 = \left(V_{1_x} \cdot V_{2_x} + V_{1_y} \cdot V_{2_y} + V_{1_z} \cdot V_{2_z}\right) = |\mathbf{V}_1| \cdot |\mathbf{V}_2| \cos\theta$$



Projections

Portion of $\vec{\mathbf{V}}_2$ in $\vec{\mathbf{V}}_1$ direction

$$= |\vec{\mathbf{V}}_2| \cos\theta = \frac{\vec{\mathbf{V}}_1 \bullet \vec{\mathbf{V}}_2}{|\vec{\mathbf{V}}_1|}$$

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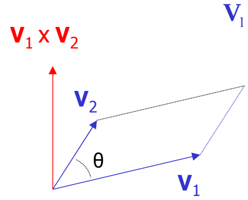
Vector Cross Product

Right-hand rule



$$\mathbf{V}_1 \times \mathbf{V}_2 = \begin{pmatrix} V_{1_y} \cdot V_{2_z} - V_{1_z} \cdot V_{2_y}, \\ V_{1_z} \cdot V_{2_x} - V_{1_x} \cdot V_{2_z}, \\ V_{1_x} \cdot V_{2_y} - V_{1_y} \cdot V_{2_x} \end{pmatrix}$$

$$= \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ V_{1_x} & V_{1_y} & V_{1_z} \\ V_{2_x} & V_{2_y} & V_{2_z} \end{vmatrix}$$

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Matrices

Rectangular matrix (m x n) (rows x cols)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Row vector $\mathbf{R} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$

Column vector $\mathbf{C} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

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Scalar Matrix Multiplication

$$\mathbf{M} = \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$$

$$a\mathbf{M} = \begin{bmatrix} au & av & aw \\ ax & ay & az \end{bmatrix}$$

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Matrix Addition

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$$

$$\mathbf{M} + \mathbf{N} = \begin{bmatrix} a+u & b+v & c+w \\ d+x & e+y & f+z \end{bmatrix}$$

Matrices must have same dimensions

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Matrix Transpose

$$\mathbf{M} = \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$$

$$\mathbf{M}^T = \begin{bmatrix} u & x \\ v & y \\ w & z \end{bmatrix}$$

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Matrix Multiplication

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{bmatrix}$$

$$\mathbf{C} = \mathbf{AB} = \left[c_{ij}\right] \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Matrices must be conformable (n=p)

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Matrix Multiplication Example

A B C

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 23 & 30 \\ 53 & 66 \end{bmatrix}$$

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 5 = 2 + 6 + 15 = 23$$

C row = A row, C column = B column

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Identity Matrix and Inverse

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Inverse computed by Gaussian elimination, determinants, or other methods; used directly or indirectly to solve sets of linear equations

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Determinants

- Only on square matrices
  - For an upper triangular matrix
    - $|\vec{\mathbf{A}}| = \prod_{k=1}^{n} a_{kk}$
- Gaussian elimination is the best method
  - Swapping two rows changes the sign of $|\vec{\mathbf{A}}|$
  - Multiplying a row by s, multiplies $|\vec{\mathbf{A}}|$ by s
  - Adding row multiples has no effect

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